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On the Numerical Solution of Two-Dimensional Potential Problems by an Improved Boundary Integral Equation Method*

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The use of piecewise quadratic polynomial approximations in the boundary integral equation method for the solution of boundary value problems involving Laplace's equation and certain Poisson equations is described. To illustrate various features of this technique the results of several numerical experiments are presented.

1. INTRODUCTION

Various numerical techniques have been proposed for the solution of elliptic boundary value problems, the most commonly used of these being finite-difference and finite-element Galerkin methods. A method of increasing popularity is the boundary integral equation (BIE) method. While the BIE method is of less general applicability than the finite-element Galerkin method it has proved to be more effective than this method for the solution of certain types of problems (see [3]).

The BIE method is based on Green's formula which enables one to reformulate certain elliptic boundary value problems as integral equations. Such an integral equation involves the solution to the problem only on the boundary of the region and also its normal derivative there, and thus the reformulation has the effect of reducing the dimension of the problem. If, for example, the given problem involves Dirichlet-type boundary conditions, the BIE method enables one to determine an approximation

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to the flux on the boundary, which may be exactly the information required from the problem. If an approximation to the value of the solution is then desired at an internal point of the region, this can be obtained by performing a simple integration which involves the specified boundary data and the boundary data obtained from the BIE method.

In the "classical" BIE method (cf. [7, 10, 19]) the boundary of the region is subdivided into arcs whose endpoints include corners of the region and points at which the form of the boundary conditions changes. The solution to the problem and its normal derivative on the boundary are then chosen to be constants on each boundary arc. In this paper we shall describe the BIE method for the solution of boundary value problems for Laplace's equation and for certain Poisson problems in the plane, devoting special attention to the case in which the approximations to the solution and the flux on the boundary of the region are generated from piecewise quadratic polynomial functions. The numerical results presented in Section 4 demonstrate that such approximations are of higher accuracy than those produced by the classical BIE method with a comparable number of boundary nodes.

The BIE method can be used to solve more general problems in the plane than those described herein (see, for example, [3, 6]). Also certain of the techniques described in this paper can be extended to three-dimensional problems (see, for example, [17]).

Even in its simplest form, the BIE method gives rise to a system of linear algebraic equations which, unlike those arising in finite-difference methods and finite-element Galerkin methods, is dense. However, by appropriately subdividing the region in which the boundary value problem is posed, sparseness can be introduced into the linear system arising from the BIE method applied to the subregions. This feature of the method, described in [7], will not be employed in the numerical experiments discussed in this paper. Nevertheless, in general, the number of nonzero elements in the coefficient matrix of the linear system arising from a finite-element Galerkin method is comparable to the number of matrix elements which have to be calculated in the BIE method with the same number of boundary nodes. Moreover, in the BIE method the elements involve only one-dimensional integrals, while those in the finite-element Galerkin method involve two-dimensional integrals. The BIE method is easy to program even for problems involving regions with curved boundaries, and is particularly effective when the solution on the boundary only or the flux on the boundary is required, or when the solution at a few interior points is desired. If an approximation is required throughout the interior of the region, the use of the BIE method is not usually recommended.

An outline of the remainder of the paper is as follows. In Section 2 the integral equation formulations of boundary value problems involving Laplace's equation and certain Poisson equations are described and the classical BIE method reviewed. The improved BIE method is presented in Section 3, and in Section 4 this method is used to obtain numerical solutions to problems chosen from the literature and some comparisons with the classical BIE method are made. In the Appendix the application of the BIE method to biharmonic problems is sketched.

2. The BIE Formulation

Suppose Ω is a bounded domain in the plane with piecewise smooth boundary $\partial\Omega$. For any sufficiently smooth function u defined on $\bar{\Omega}$, the closure of Ω, it is well known from Green's third identity [2, pp. 256–257], that, for $P \in \bar{\Omega}$,

$$\int_{\partial\Omega} \left[u(Q) \frac{\partial}{\partial n_Q} \log r(P, Q) - \frac{\partial u(Q)}{\partial n_Q} \log r(P, Q)\right] ds(Q)$$

$$+ \int_{\Omega} \nabla_q^2 u(q) \log r(P, q)\, dA(q) = c(P)\, u(P), \tag{2.1}$$

where $Q \in \partial\Omega$, $q \in \Omega$, ∇^2 denotes the Laplace operator at the point q, n_Q denotes the outward normal to $\partial\Omega$ at Q, $r(P, Q)$ (resp. $r(P, q)$) denotes the distance between the points P and Q (resp. P and q), and

$$c(P) = \int_{\partial\Omega} \frac{\partial}{\partial n_Q} \log r(P, Q)\, ds(Q).$$

Since

$$\frac{\partial}{\partial n_Q} \log r(P, Q) = \frac{d\theta}{ds}(P, Q), \tag{2.2}$$

where

$$\theta(P, Q) = \tan^{-1}\left(\frac{y(Q) - y(P)}{x(Q) - x(P)}\right),$$

it follows that

$$c(P) = \begin{cases} 2\pi, & P \in \Omega, \\ 0, & P \notin \bar{\Omega}, \end{cases}$$

and if $P \in \partial\Omega$ and $\partial\Omega$ has a unique tangent at P, then $c(P) = \pi$.

Now suppose that

$$\nabla^2 u(P) = 0, \qquad P \in \Omega.$$

Then from (2.1) we have

$$c(P)\, u(P) = \int_{\partial\Omega} \left\{u(Q) \frac{\partial}{\partial n_Q} \log r(P, Q) - \frac{\partial u(Q)}{\partial n_Q} \log r(P, Q)\right\} ds(Q). \tag{2.3}$$

If u is prescribed on $\partial\Omega$, that is, we have a Dirichlet problem, then (2.3) with $P \in \partial\Omega$ becomes a Fredholm integral equation of the first kind for the unknown boundary flux $\partial u/\partial n_Q$. In many Dirichlet problems, this boundary flux is the desired information. However, if the value of the solution u is required at a point $P \in \Omega$, this can be obtained from (2.3) using the calculated value of the boundary flux and the prescribed values of u on $\partial\Omega$. It should be noted that, while the solution u of the Dirichlet problem may be unique, there are exceptional cases in which the corresponding Fredholm integral equation of the first kind does not have a unique solution $\partial u/\partial n_Q$. This phenomenon is discussed, for example, in [1], where it is shown that in such exceptional cases a supplementary condition must be imposed on $\partial u/\partial n_Q$ in order to obtain a unique solution to the integral equation. This condition, which takes the form

$$\int_{\partial\Omega} \frac{\partial u}{\partial n_Q}\, ds(Q) = 0, \tag{2.4}$$

must be satisfied in general by harmonic functions since, using Gauss' theorem,

$$\int_{\partial\Omega} \frac{\partial u}{\partial n_Q}\, ds(Q) = \int_{\Omega} \nabla_q^2 u(q)\, dA(q).$$

The use of the classical BIE method incorporating condition (2.4) for the solution of Laplace's equation in exceptional cases is also discussed in [1]. Condition (2.4) in a suitably discretized form can be easily incorporated into the improved BIE method.

If we have a Neumann problem and $\partial u/\partial n_Q$ is prescribed on the boundary so that

$$\int_{\partial\Omega} \frac{\partial u}{\partial n_Q}\, ds(Q) = 0 \tag{2.5}$$

(a necessary condition for the existence of a solution u, not to be confused with condition (2.4) in which $\partial u/\partial n_Q$ is *unknown*), then (2.3) with $P \in \partial\Omega$ yields a Fredholm integral equation of the second kind for the (nonunique) values of u on $\partial\Omega$. To obtain a unique solution an additional restriction must be imposed on u such as its value at some point of $\partial\Omega$ or

$$\int_{\partial\Omega} u(Q)\, ds(Q) = 0. \tag{2.6}$$

Having solved this integral equation, the values of u at any point in Ω can again be determined from (2.3).

Other types of boundary conditions are also easily handled. In the case of boundary conditions of the form

$$\alpha u + \beta \frac{\partial u}{\partial n} = g, \qquad \alpha\beta \neq 0, \tag{2.7}$$

where α, β, and g are given functions, we substitute for $\partial u/\partial n$ in (2.3) and, for $P \in \partial\Omega$, again obtain a Fredholm integral equation of the second kind for u on $\partial\Omega$. When mixed boundary conditions are prescribed, the given data are inserted into (2.3) with $P \in \partial\Omega$, yielding an integral equation for the unknown boundary data. Note that when Neumann data are prescribed on only part of the boundary, restriction (2.6) is no longer required.

If the equation under consideration is Poisson's equation

$$\nabla^2 u = f \qquad \text{in } \Omega, \tag{2.8}$$

then (2.1) contains an integral over Ω, namely,

$$\int_{\Omega} f(q) \log r(P, q)\, dA(q).$$

If f is harmonic in Ω this term can be reformulated as an integral over $\partial\Omega$. In fact it is easy to show, using Green's second identity [2, p. 252], that, for $P \in \bar{\Omega}$,

$$\begin{aligned} &\int_{\Omega} f(q) \log r(P, q)\, dA(q) \\ &\quad = \frac{1}{4} \int_{\partial\Omega} \left[f(Q) \frac{\partial}{\partial n_Q} G(P, Q) - \frac{\partial f(Q)}{\partial n_Q} G(P, Q) \right] ds(Q), \end{aligned} \tag{2.9}$$

where $G(P, Q) = [r(P, Q)]^2[\log r(P, Q) - 1]$. Thus, for $P \in \partial\Omega$, (2.1) can be written in the form

$$\begin{aligned} c(P)\, u(P) &- \int_{\partial\Omega} \left\{ u(Q) \frac{\partial}{\partial n_Q} \log r(P, Q) - \frac{\partial u(Q)}{\partial n_Q} \log r(P, Q) \right\} ds(Q) \\ &= \frac{1}{4} \int_{\partial\Omega} \left[f(Q) \frac{\partial}{\partial n_Q} G(P, Q) - \frac{\partial f(Q)}{\partial n_Q} G(P, Q) \right] ds(Q) \\ &\equiv F(P), \end{aligned} \tag{2.10}$$

in which all the integrals are integrals on the boundary of Ω.

In general it is, of course, impossible to solve the boundary integral equations exactly, and hence one must resort to a numerical technique. In the classical BIE method ([7, 10, 19]) the boundary $\partial\Omega$ is subdivided into arcs $\partial\Omega_j \equiv \overset{\frown}{P_{j-1}P_j}$, $j = 1,\ldots, M$, where $P_M = P_0$. On each arc $\partial\Omega_j$, u and $\partial u/\partial n$ are chosen to be constants, u_j and $u_{n,j}$, say, respectively. Then, for example, (2.3) becomes

$$c(P)\, u_i = \sum_{j=1}^{M} \left\{ u_j \int_{\partial\Omega_j} \frac{\partial}{\partial n_Q} \log r(P, Q)\, ds(Q) - u_{n,j} \int_{\partial\Omega_j} \log r(P, Q)\, ds(Q) \right\}, \tag{2.11}$$

where $P \in \partial\Omega_i$. If we collocate Eq. (2.11) at the midpoints $\hat{P}_i$ of the arcs $\partial\Omega_i$, $i = 1,\ldots, M$, we obtain the system of linear algebraic equations

$$AU = BU_n, \tag{2.12}$$

where

$$U = (u_1, \ldots, u_M)^T$$

and

$$U_n = (u_{n,1}, \ldots, u_{n,M})^T.$$

Also $A = (a_{ij})$ and $B = (b_{ij})$ where

$$a_{ij} = \int_{\partial\Omega_j} \frac{\partial}{\partial n_Q} \log r(\hat{P}_i, Q)\, ds(Q) - \delta_{ij} c(\hat{P}_i),$$

with δ_{ij} denoting the Kronecker delta, and

$$b_{ij} = \int_{\partial\Omega_j} \log r(\hat{P}_i, Q)\, ds(Q).$$

In the case of Dirichlet boundary conditions, one would choose for u_j the value $u(\hat{P}_j)$ and solve (2.12) for U_n. When Neumann boundary conditions are prescribed, set $u_{nj} = (\partial u/\partial n)(\hat{P}_j)$ and replace (2.6) by an appropriate approximation, for example,

$$\sum_{j=1}^{M} u_j \int_{\partial\Omega_j} ds(Q) = 0, \tag{2.13}$$

and then determine the least-squares solution of the $M + 1$ equations (2.12) and (2.13). In other types of boundary conditions u and $\partial u/\partial n$ are approximated by their values at the midpoints of the arcs on which they are specified. The approximations to these functions on the remaining segments are determined from (2.12).

If $\partial\Omega$ is curved, it may be desirable to approximate $\partial\Omega_j$ by the segments $\overline{P_{j-1}\hat{P}_j}$ and $\overline{P_j\hat{P}_j}$, and integrate over these segments exactly [20] or use an appropriate quadrature formula. An alternate procedure is to replace $\partial\Omega_j$ by the segment $\overline{P_{j-1}P_j}$, evaluate the given data at the midpoint of this segment, and perform all integrations over this segment.

3. Piecewise Polynomial Approximations in the BIE Method

The numerical procedure outlined in Section 2 has proved to be quite effective in practice (see [15, 16] for example). However, more accurate approximations to the solutions of the boundary integral equations can be obtained using isoparametric-type techniques similar to those used in the finite-element Galerkin method (cf. [22]). In this approach, on each arc $\partial\Omega_j$ of the boundary $\partial\Omega$, u and $\partial u/\partial n$ and possibly $\partial\Omega_j$ itself are approximated using polynomials. We shall describe in detail the case in which quadratics are employed.

In the following we shall denote by $P_{1,j}$ and $P_{3,j}$ the end points of $\partial\Omega_j$, and by $P_{2,j}$ its midpoint. Note that $P_{3,k} = P_{1,k+1}$, $k = 1,\ldots, M-1$, and $P_{3,M} = P_{1,1}$. For $-1 \leqslant \xi \leqslant 1$, let

$$\begin{aligned} M_1(\xi) &= -\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^2, \\ M_2(\xi) &= 1 - \xi^2, \\ M_3(\xi) &= \tfrac{1}{2}\xi + \tfrac{1}{2}\xi^2, \end{aligned} \tag{3.1}$$

and consider the transformation defined by

$$x(\xi) = \sum_{i=1}^{3} M_i(\xi)\, x_i^j, \qquad y(\xi) = \sum_{i=1}^{3} M_i(\xi)\, y_i^j, \tag{3.2}$$

where (x_i^j, y_i^j) are the Cartesian coordinates of the point $P_{i,j}$. This transformation maps the points $P_{i,j}$, $i = 1, 2, 3$ onto the points $-1, 0, 1$, respectively, on the ξ-axis. If $Q = (x, y) \in \partial\Omega_j$, then we approximate $u(Q)$ by

$$\tilde{u}(Q) = \sum_{i=1}^{3} \tilde{u}(P_{i,j})\, M_i(\xi)$$

and $\partial u(Q)/\partial n_Q$ by

$$\tilde{u}_n(Q) = \sum_{i=1}^{3} \tilde{u}_n(P_{i,j})\, M_i(\xi).$$

If u (resp. $\partial u/\partial n$) is prescribed on $\partial\Omega_j$ then $\tilde{u}(P_{i,j})$ (resp. $\tilde{u}_n(P_{i,j})$) is taken to be $u(P_{i,j})$ (resp. $(\partial u/\partial n)(P_{i,j})$). On substituting these approximations into (2.3) and collocating the resulting equation at each of the boundary nodes $P_{1,j}$ and $P_{2,j}$ ($j = 1,\ldots, M$), we obtain the system of linear algebraic equations

$$c(P)\,\tilde{u}(P) = \sum_{j=1}^{M} \left\{ \sum_{i=1}^{3} \tilde{u}(P_{i,j}) \int_{\partial\Omega_j} M_i(\xi) \frac{\partial}{\partial n_Q} \log r(P, Q)\, ds(Q) - \sum_{i=1}^{3} \tilde{u}_n(P_{i,j}) \int_{\partial\Omega_j} M_i(\xi) \log r(P, Q)\, ds(Q) \right\}, \tag{3.3}$$

where $P \in \mathscr{P} \equiv \{P_{l,k},\ l = 1, 2;\ k = 1,\ldots, M\}$. When $\partial\Omega$ is curved, the integrals over the arcs $\partial\Omega_j$, $j = 1,\ldots, M$, may be complicated, in which case the transformation (3.2) is used to approximate the integrals over $\partial\Omega_j$ by integrals over $[-1, 1]$, and (3.3) becomes, after the use of (2.2) and a little rearranging,

$$\sum_{j=1}^{M} \left[\sum_{i=1}^{3} \tilde{u}(P_{i,j})\, \alpha_{i,j}(P) - \tilde{u}(P)\, \alpha_j(P) \right] = \sum_{j=1}^{M} \sum_{i=1}^{3} \tilde{u}_n(P_{i,j})\, \beta_{i,j}(P), \tag{3.4}$$

where $P \in \mathscr{P}$,

$$\alpha_{i,j}(P) = \int_{-1}^{1} M_i(\xi) \frac{d\theta(P, Q(\xi))}{d\xi}\, d\xi, \qquad \alpha_j(P) = \int_{-1}^{1} \frac{d\theta(P, Q(\xi))}{d\xi}\, d\xi, \tag{3.5}$$

and

$$\beta_{i,j}(P) = \int_{-1}^{1} M_i(\xi) \log r(P, Q(\xi))\, J_j(\xi)\, d\xi, \tag{3.6}$$

where

$$J_j(\xi) = \left[\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2\right]^{1/2}$$

$$= \left[\left(\sum_{i=1}^{3} M_i' x_i^j\right)^2 + \left(\sum_{i=1}^{3} M_i' y_i^j\right)^2\right]^{1/2}.$$

If $\tilde{U}$ denotes the $2M$-vector with components

$$\tilde{U}_{2N-1} = \tilde{u}(P_{1,N}), \qquad N = 1,\ldots, M,$$
$$\tilde{U}_{2N} = \tilde{u}(P_{2,N}), \qquad N = 1,\ldots, M,$$

and $\tilde{U}_n$ is defined similarly in terms of $\tilde{u}_n(P_{1,N})$ and $\tilde{u}_n(P_{2,N})$, then (3.4) can be written in the matrix form

$$\tilde{A}\tilde{U} = \tilde{B}\tilde{U}_n, \tag{3.7}$$

where $\tilde{A} = (\tilde{a}_{m,n})$ and $\tilde{B} = (\tilde{b}_{m,n})$ with

$$\begin{aligned} \tilde{a}_{m,2N-1} &= \alpha_{3,N-1}(P_{l,k}) + \alpha_{1,N}(P_{l,k}) - \delta_{m,2N-1} \sum_{j=1}^{M} \alpha_j(P_{l,k}), \\ \tilde{a}_{m,2N} &= \alpha_{2,N}(P_{l,k}) - \delta_{m,2N} \sum_{j=1}^{M} \alpha_j(P_{l,k}) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \tilde{b}_{m,2N-1} &= \beta_{3,N-1}(P_{l,k}) + \beta_{1,N}(P_{l,k}), \\ \tilde{b}_{m,2N} &= \beta_{2,N}(P_{l,k}), \end{aligned} \tag{3.9}$$

where $m = 2k + l - 2$, $l = 1, 2$, and $k = 1,\ldots, M$. In (3.8) and (3.9) we have used the fact that $P_{3,k-1} = P_{1,k}$ and $P_{3,0} = P_{3,M}$.

In the construction of the matrices $\tilde{A}$ and $\tilde{B}$ we have assumed that both u and $\partial u/\partial n$ are continuous on the boundary $\partial\Omega$. The right-hand side of (3.7) can be easily modified to take into account jump discontinuities in $\partial u/\partial n$ which may occur at the nodes $P_{1,j}$, $j = 1,\ldots, M$. In problems involving boundary singularities or discontinuous Dirichlet boundary conditions, the use of the improved BIE method as described would be inadvisable, since it is clearly inappropriate to approximate $\partial u/\partial n$ by a polynomial near a point at which u has singular behavior. Papamichael and Symm [14] have described several ways in which the classical BIE method can be modified to treat such problems. Similar modifications can be devised for the improved BIE method but will not be discussed herein.

The potential advantages of the use of piecewise polynomial approximations in the BIE method will be lost if the integrals in (3.8) and (3.9) are not carefully treated.

Consider first (3.8). If $P = P_{l,k}$, $l = 1$ or 2, $1 \leqslant k \leqslant M$, then $\alpha_{i,j}(P_{l,k})$, $\alpha_j(P_{l,k})$, and $\beta_{i,j}(P_{l,k})$ have well-behaved integrands when $P \notin \partial\Omega_j$, and these integrals can be approximated using an appropriate Gaussian quadrature formula. (In practice a four-point formula is used.) When $P \in \partial\Omega_j$,

$$\begin{aligned}\alpha_{i,j}(P) &- \delta_{il}\alpha_j(P)\\ &= \int_{-1}^{1} (M_i(\xi) - \delta_{il}) \frac{d\theta}{d\xi}\, d\xi\\ &= \int_{-1}^{1} (M_i(\xi) - \delta_{il}) \frac{y'(\xi)(x(\xi) - x_l^j) - x'(\xi)(y(\xi) - y_l^j)}{(x(\xi) - x_l^j)^2 + (y(\xi) - y_l^j)^2}\, d\xi. \qquad (3.10)\end{aligned}$$

The integrand in this case has a removable singularity at the point $\xi(P_{l,k})$. Gaussian quadrature can again be used to approximate this integral. Thus the elements of the matrix $\tilde{A}$ can be formed by evaluating the integrals in (3.8) individually except when $P \in \partial\Omega_j$. In that case certain of these integrals must be grouped as indicated in (3.10). Such groupings are necessary only in the computation of the diagonal elements of $\tilde{A}$, one being required in $\tilde{a}_{2N,2N}$, and two in $\tilde{a}_{2N-1,2N-1}$ since $P_{3,N-1} = P_{1,N}$ ($P_{3,0} = P_{3,M}$) and we compute $\alpha_{3,N-1}(P_{1,N}) - \alpha_{N-1}(P_{1,N})$ and $\alpha_{1,N}(P_{1,N}) - \alpha_N(P_{1,N})$.

Consider now the evaluation of $\beta_{i,j}(P)$ when $P = P_{l,j}$, $l = 1$, 2, or 3. If $l = 1$, then

$$\begin{aligned}\beta_{i,j}(P) &= \int_{-1}^{1} M_i(\xi) \log r(P_{1,j}, Q(\xi))\, J_j(\xi)\, d\xi\\ &= \int_{-1}^{1} M_i(\xi) \log[2r(P_{1,j}, Q(\xi))/(1 + \xi)]\, J_j(\xi)\, d\xi\\ &\quad + \int_{-1}^{1} M_i(\xi) \log[(1 + \xi)/2]\, J_j(\xi)\, d\xi.\end{aligned}$$

The first integral has a removable singularity at $\xi = -1$ and can be evaluated using Gaussian quadrature. The transformation

$$\zeta = \tfrac{1}{2}(1 + \xi)$$

transforms the second into

$$2 \int_{0}^{1} M_i(2\zeta - 1) \log \zeta J_j(2\zeta - 1)\, d\zeta$$

which can be evaluated using a weighted Gaussian quadrature formula with weight $\log \zeta$ (see [12] or [21]). If $l = 3$, the procedure is similar but with $\frac{1}{2}(1 - \xi)$ replacing $\frac{1}{2}(1 + \xi)$. When $l = 2$,

$$\begin{aligned}\beta_{i,j}(P) &= \int_{-1}^{0} M_i(\xi) \log[r(P_{2,j}, Q(\xi)]/(-\xi)]\, J_j(\xi)\, d\xi\\ &\quad + \int_{0}^{1} M_i(\xi) \log[r(P_{2,j}, Q(\xi))/\xi]\, J_j(\xi)\, d\xi + \int_{-1}^{0} M_i(\xi) \log(-\xi)\, J_j(\xi)\, d\xi\\ &\quad + \int_{0}^{1} M_i(\xi) \log(\xi)\, J_j(\xi)\, d\xi.\end{aligned}$$

The first two integrals have removable singularities at $\xi = 0$ and can be approximated using a Gaussian quadrature formula. A log ξ-weighted Gaussian formula can be used to evaluate the remaining integrals.

If we have a Poisson problem leading to (2.9), then $F(P)$ of (2.10) is evaluated by approximating f and $\partial f/\partial n$ on $\partial\Omega_j$, $j = 1,\ldots, M$, by

$$\tilde{f}(Q) = \sum_{i=1}^{3} f(P_{i,j})\, M_i(\xi)$$

and

$$\tilde{f}_n(Q) = \sum_{i=1}^{3} \frac{\partial f}{\partial n}(P_{i,j})\, M_i(\xi)$$

and then transforming the integrals over $\partial\Omega_j$ to integrals over $[-1, 1]$. The latter are well behaved and can be evaluated using an appropriate Gaussian quadrature formula.

4. Numerical Results

In this section we present numerical results which illustrate the performance of the improved BIE method (IBIEM) defined by (3.3) on three test problems chosen from the literature. The first two problems involve Laplace's equation, one with Dirichlet boundary conditions, the other having mixed boundary conditions. The third problem involves Poisson's equation subject to Dirichlet boundary conditions. In two of the problems the regions have curved boundaries. The test problems cover most of the cases described in this paper and help demonstrate the versatility as well as the accuracy of the IBIEM.

The systems of linear algebraic equations (3.7) arising in the IBIEM were solved using the routine MA21A from the Harwell Subroutine Library, and all computations were performed on the IBM 370–165 at the University of Kentucky's Computer Center.

Problem 1. Consider the Dirichlet problem

$$\begin{aligned} \nabla^2 u(x, y) &= 0, && (x, y) \in \Omega, \\ u(x, y) &= -x^3 - 3x^2y + 3xy^2 + y^3, && (x, y) \in \partial\Omega, \end{aligned}$$

where

$$\bar{\Omega} \equiv \{(x, y)\colon x^2 + y^2 \leqslant 1, x \geqslant 0, y \geqslant 0\}.$$

the analytical solution of which is

$$u(x, y) = -x^3 - 3x^2y + 3xy^2 + y^3.$$

This problem was solved in [5] using finite differences and the method of successive overrelaxation, and in [14] using the classical BIE method (2.11) (denoted in the following by CBIEM) and modifications of it.

Approximations to u at certain points of Ω were generated from the IBIEM with $M = 14, 28, 58$, where M denotes the number of boundary nodes. The number of segments $\overline{M}$ on each part of the boundary is given in Table I. In this and following problems, unless otherwise stated, the segments on each portion of the boundary are of equal length. In Table II we have presented the errors (analytical solution minus numerical solution) in approximations to u at the points of Ω indicated in Fig. 1. For comparison we have given in Table II the results obtained using the CBIEM with $M = 29$ and the same segment distribution as for the IBIEM with $M = 58$. Also quoted are results given in [14] for the CBIEM with $M = 84$, 24 equal segments on each straight edge and 36 on the curved boundary.

TABLE I

Distribution of Segments (Problem 1)

M \ $\overline{M}$	$y = 0$	$x = 0$	$x^2 + y^2 = 1$
14	2	2	3
28	4	4	6
58	8	8	13

TABLE II

Errors $\times 10^4$ in BIEM Solutions (Problem 1)

Point	Analytical solution	IBIEM $M = 14$	IBIEM $M = 28$	IBIEM $M = 58$	CBIEM $M = 29$	CBIEM $M = 84$
1	0.0000	0	0	0	0	0
2	0.0440	−10	0	0	−3	0
3	0.1840	−4	−4	0	−5	−1
4	0.4680	34	0	0	−1	0
5	0.9440	112	12	0	20	1
6	0.0000	0	0	0	0	0
7	0.1880	5	0	0	3	0
8	0.5680	10	2	0	16	1
9	1.1880	−813	5	−4	22	3
10	0.0000	0	0	0	0	0
11	0.4280	79	2	1	17	1

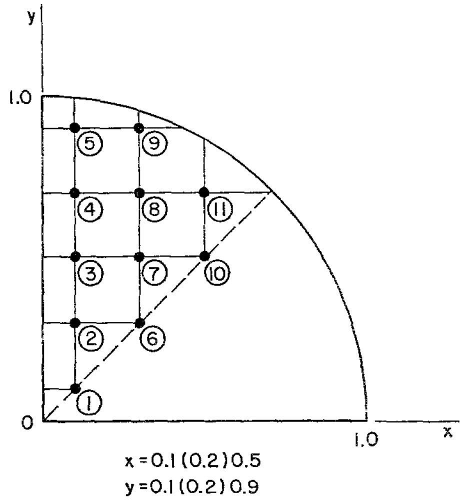

FIG. 1. Region of Problem 1.

PROBLEM 2. This problem was taken from [11], where it was used as a test problem for the method of lines. In this problem

$$\bar{\Omega} \equiv \{(x, y)\colon 0 \leqslant x \leqslant \tfrac{1}{2}, 0 \leqslant y \leqslant b\}$$

and

$$\nabla^2 u(x, y) = 0, \qquad (x, y) \in \Omega,$$

with

$$\begin{aligned} u(0, y) &= 0, & 0 \leqslant y \leqslant b, \\ \frac{\partial u}{\partial x}(\tfrac{1}{2}, y) &= 0, & 0 \leqslant y \leqslant b, \\ \frac{\partial u}{\partial y}(x, 0) &= 0, & 0 \leqslant x \leqslant \tfrac{1}{2}, \\ u(x, b) &= \sin \pi x, & 0 \leqslant x \leqslant \tfrac{1}{2}. \end{aligned}$$

The analytical solution is

$$u(x, y) = \frac{\cosh \pi y \sin \pi x}{\cosh \pi b},$$

and, as in [11], we chose $b = 0.475$. Approximations to $\partial u/\partial n$ on the lines $x = 0$ and $y = b$ and to u on the lines $y = 0$ and $x = \frac{1}{2}$ were calculated using the IBIEM with $M = 8$, 16, and 32, and 1, 2, and 4 segments per side, respectively. The errors in these approximations at the boundary points indicated in Fig. 2 are given in Table III. For comparison purposes, the values obtained from the CBIEM with $M = 16$ (4 equal segments per side) are also stated.

In this and the following example approximations at boundary points which are not nodes are simply obtained from the assumed form of the approximations on the boundary.

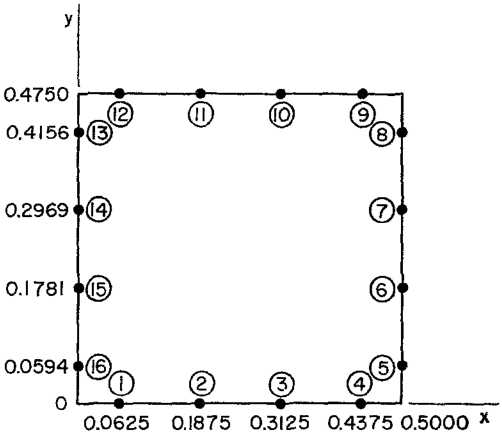

FIG. 2. Region of Problem 2.

TABLE III

Errors × 10^4 in BIEM Solutions (Problem 2)

Point	Analytical solution	IBIEM $M = 8$	IBIEM $M = 16$	IBIEM $M = 32$	CBIEM $M = 16$
1	0.0835	−82	−15	−1	49
2	0.2378	−45	15	0	30
3	0.3559	70	−5	0	16
4	0.4199	83	8	1	6
5	0.4356	93	8	1	22
6	0.4969	94	−3	1	5
7	0.6282	−27	24	1	−13
8	0.8479	−85	−17	2	−28
9	2.7846	77	47	24	−1536
10	2.3607	344	−13	9	167
11	1.5774	−240	100	8	−239
12	0.5539	−485	−82	24	−731
13	−2.6637	−285	36	−18	3661
14	−1.9734	318	−37	−9	−672
15	−1.5610	−10	15	−6	−431
16	−1.3683	−115	−17	−3	590

PROBLEM 3. This problem, which involves Poisson's equation, was used in [23] as a test problem for various finite-element Galerkin methods. It takes the form

$$\nabla^2 u(x, y) = -\frac{26}{3} xy, \qquad (x, y) \in \Omega,$$

$$u(x, y) = 0, \qquad (x, y) \in \partial\Omega,$$

where

$$\bar{\Omega} \equiv \{(x, y): \tfrac{4}{9}x^2 + y^2 \leqslant 1, x \geqslant 0, y \geqslant 0\},$$

and its analytical solution is

$$u(x, y) = xy(1 - \tfrac{4}{9}x^2 - y^2).$$

Approximations to $\partial u/\partial n$ were obtained using the IBIEM based on (2.10) with $M = 12$, 20, and 38, the distribution of segments on each portion of the boundary being given in Table IV. In this problem the segments on the curved portion of the boundary were not of equal length. The errors in the approximations to $\partial u/\partial n$ at the points on $\partial\Omega$ indicated in Fig. 3 are given in Table V.

TABLE IV

Distribution of Segments (Problem 3)

M \ $\bar{M}$	$y = 0$	$x = 0$	$\frac{4}{9}x^2 + y^2 = 1$
12	2	1	3
20	3	2	5
38	6	4	10

TABLE V

Errors $\times\ 10^4$ in BIEM Solutions (Problem 3)

Point	Analytical solution	IBIEM $M = 12$	IBIEM $M = 20$	IBIEM $M = 38$
1	−0.2431	40	5	0
2	−0.5625	−1	−3	−1
3	−0.3819	−31	1	0
4	−0.5027	1	−13	−1
5	−1.2242	251	−18	9
6	−1.2493	189	−7	−6
7	−0.8941	15	1	2
8	−0.3218	147	13	−1
9	−0.3281	−213	−2	1
10	−0.2348	403	−1	−4

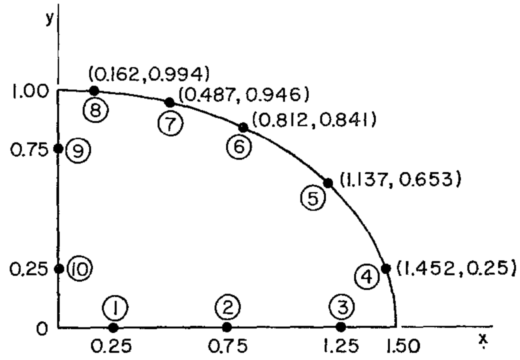

FIG. 3. Region of Problem 3.

APPENDIX. THE BIE METHOD FOR THE TWO-DIMENSIONAL BIHARMONIC EQUATION

The techniques described in this paper can also be used to solve boundary value problems for the two-dimensional biharmonic equation.

$$\nabla^4 u(P) = 0, \qquad P \in \Omega, \tag{A1a}$$

where Ω is a bounded domain in the plane with boundary $\partial\Omega$. Of particular interest are Dirichlet boundary conditions of the form

$$u(P) = f(P), \qquad \frac{\partial u(P)}{\partial n_P} = g(P), \qquad P \in \partial\Omega. \tag{A1b}$$

Under sufficient smoothness assumptions on f and g there exists a unique solution of (A1). By introducing $v = \nabla^2 u$, (A1) can be written as the coupled system

$$\nabla^2 u(P) = v(P), \qquad P \in \bar{\Omega}, \tag{A2a}$$

$$\nabla^2 v(P) = 0, \qquad P \in \Omega, \tag{A2b}$$

$$u(P) = f(P), \qquad \frac{\partial u(P)}{\partial n} = g(P), \qquad P \in \partial\Omega. \tag{A2c}$$

To solve (A2) we first express it as a system of coupled integral equations on $\partial\Omega$. Since v is harmonic, it follows from (2.10) that

$$\begin{aligned}\frac{1}{4}\int_{\partial\Omega}&\left[v(Q)\frac{\partial}{\partial n_Q}G(P,Q) - \frac{\partial v(Q)}{\partial n_Q}G(P,Q)\right] ds(Q)\\ &= c(P)\,u(P) - \int_{\partial\Omega}\left[u(Q)\frac{\partial}{\partial n_Q}\log r(P,Q)\right.\\ &\qquad \left.- \frac{\partial}{\partial n_Q}u(Q)\log r(P,Q)\right] ds(Q), \qquad P \in \partial\Omega,\end{aligned} \tag{A3}$$

and from (2.3) that

$$c(P)\,v(P) - \int_{\partial\Omega} \left[v(Q) \frac{\partial}{\partial n_Q} \log r(P, Q) - \frac{\partial v(Q)}{\partial n_Q} \log r(P, Q) \right] ds(Q) = 0, \qquad P \in \partial\Omega. \tag{A4}$$

Since u satisfies the boundary conditions (A2c) the right-hand side of (A3) is known, and Eqs. (A3) and (A4) form a coupled pair of integral equations foı the functions v and $\partial v/\partial n$ on $\partial\Omega$. Approximations to these functions can be determined using quadratic functions as described in this paper. Since these approximations must be calculated simultaneously one must solve a system of equations of twice the order of that occurring in the solution of Laplace's equation using the same number of boundary nodes.

Once approximations to $v(P)$ and $\partial v(P)/\partial n_P$, $P \in \partial\Omega$, have been determined, an approximation to $u(P)$, $P \in \Omega$, can be calculated from (A3) with $P \in \Omega$.

Finite-difference methods for the solution of (A2) are discussed in [4], for example, for the case in which Ω is a rectangle. One difficulty encountered in such methods is that since v $(= \nabla^2 u)$ is unknown on $\partial\Omega$, special methods for approximating v on Ω must be devised. Note that in the BIE formulation of (A2) the fact that boundary values for v are not prescribed gives rise to no new difficulties.

When boundary conditions such as

$$u = f(P), \qquad \nabla^2 u(P) = g(P), \qquad P \in \partial\Omega$$

(or

$$u = f(P), \qquad \frac{\partial}{\partial n_P} \nabla^2 u(P) = g(P), \qquad P \in \partial\Omega)$$

are prescribed then the problem is much simpler since (A2a) and (A2b), and hence the integral equations (A3) and (A4) are weakly coupled. One can determine an approximation to $\partial v(P)/\partial n_P$, $P \in \partial\Omega$ (or $v(P)$, $P \in \partial\Omega$) using (A4) and then using (A3) an approximation to $u(P)$, $P \in \partial\Omega$. Classical BIE methods for approximating solutions to such problems are discussed in [8, 9, 13, 18].

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